DISCRETE SUBSETS OF TOTALLY IMAGINARY QUARTIC ALGEBRAIC INTEGERS IN THE COMPLEX PLANE

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Abstract. Algebraic integers in totally imaginary quartic number fields are not discrete in the complex plane under a fixed embedding, which makes it impossible to visualize all integers in the plane, unlike the quadratic imaginary algebraic integers. In this note we consider a naturally occurring discrete subset of the algebraic integers with similar properties as lattices. For the fifth cyclotomic field, we investigate those integers with absolute values under a fixed embedding in a given bound. We show that such integers form a discrete set in the complex plane. It is observed that this subset has quasi-periodic appearance. In particular, we also show that the distance between a fixed point to the most adjacent point in this subset takes only two possible values.

1. Introduction

Let K be a totally imaginary quartic number field, and denote by $\mathcal{O}_K$ its ring of integers. K has a unique maximal real subfield $K_0 = K \cap \mathbb{R}$. Let σ denote the non-trivial automorphism of K_0, then σ can be extended to two embeddings of $K \hookrightarrow \mathbb{C}$. By abuse of notation we denote a fixed one of the embeddings by σ and the other embedding is then the complex conjugate of σ. For $z \in \mathcal{O}_K \subseteq \mathbb{C}$, we denote $\sigma(z) = z^\sigma$ the image under the embedding σ.

Let $\mathcal{B}$ be a bounded subset of $\mathbb{C}$ containing 0 as an interior point. Consider the set

$$\mathcal{S}_\mathcal{B} := \{z \in \mathcal{O}_K | z^\sigma \in \mathcal{B}\}.$$

We claim that the set $\mathcal{S}_\mathcal{B}$ is a discrete subset of $\mathbb{C}$.

Proposition 1.1. *Suppose $z_1, z_2 \in \mathcal{S}_\mathcal{B}$, then $|z_1 - z_2| \leq \frac{1}{2\,\mathrm{diam}(\mathcal{B})}$.*

Proof. Since $z_1, z_2 \in \mathcal{S}_\mathcal{B}$, we have

$$\mathbf{N}_{K/\mathbb{Q}}(z_1 - z_2) = |z_1 - z_2|^2 \left|z_1^\sigma - z_2^\sigma\right|^2 \leq 4\,\mathrm{diam}(\mathcal{B})^2 \cdot |z_1 - z_2|^2 .$$

On the other hand, as z_1 and z_2 are algebraic integers, we have $\mathbf{N}_{K/\mathbb{Q}}(z_1 - z_2) \geq 1$. Hence we obtain

$$|z_1 - z_2| \geq \frac{1}{2\,\mathrm{diam}(\mathcal{B})}.$$

$\square$

2. A Discrete Subset of Integers in $\mathbb{Q}(\zeta_5)$

Consider the fifth cyclotomic field $K = \mathbb{Q}(\zeta_5)$ and its ring of integers $\mathcal{O}_K$. Fix the embedding σ of K that sends $\zeta_5 = \exp(2\pi i/5)$ to ζ_5^2. Let $\mathcal{B}$ be the unit circle in $\mathbb{C}$, and consider the discrete subset $\mathcal{S} = \mathcal{S}_\mathcal{B}$. We claim that $\mathcal{S}$ has the five-fold symmetry.

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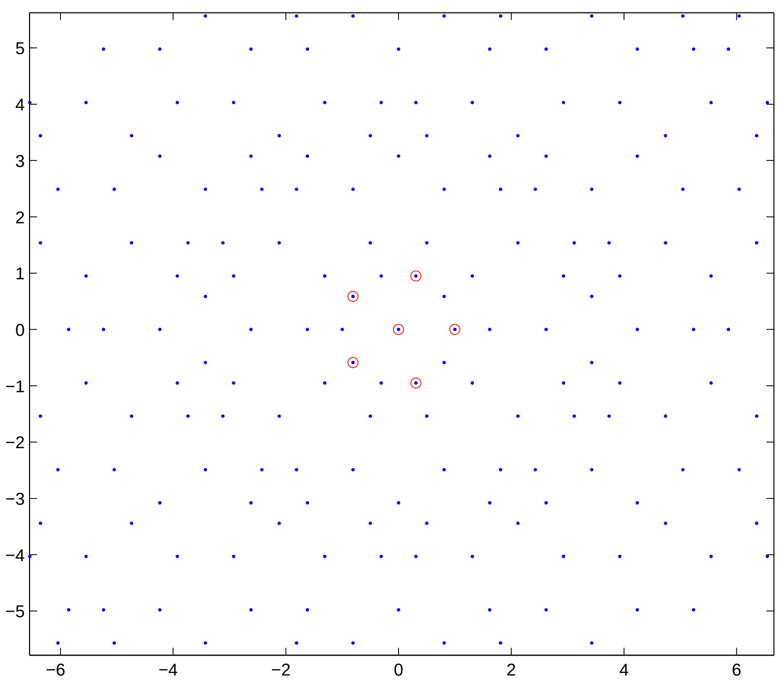
Lemma 2.1. *If $z \in \mathcal{S}$, then $\zeta z \in \mathcal{S}$.*

Proof. It suffices to observe that $|\zeta| = 1$ and hence $|\zeta z| = |z|$ for ζ any root of unity. $\square$

Note that $\mathcal{S}$ is a subset of $\mathcal{O}_K$, and S is in fact not a lattice in $\mathbb{C}$. However, $\mathcal{S}$ shares the following similar property as lattices in $\mathbb{C}$. The following figure depicts a portion of the set $\mathcal{S} \subseteq \mathbb{C}$ near the origin. The points surrounded by small circles are 0 and the fifth roots of unity.



For a fixed point z in a given lattice $\Lambda \in \mathbb{C}$, the minimum distance from z to another point $z' \in \Lambda$ is always a constant. Namely, if $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$, then $\min_{z' \in \Lambda - \{z\}} |z' - z| = \min_{z' \in \Lambda - \{0\}} |z'|$. For the set S, we have the following theorem.

Theorem 2.2. $\min_{z' \in \mathcal{S} - \{z\}} |z' - z| \in \left\{ \frac{\sqrt{5}-1}{2}, 1 \right\}$.

To prove the above theorem, we need the following lemmas.

Lemma 2.3. *Suppose $z_1, z_2 \in S$. If $|z_1 - z_2| < \frac{\sqrt{5}}{2}$, then $z_1 - z_2$ is a unit.*

Proof. Suppose to the contrary that $z_1 - z_2$ is not a unit, then $\mathbf{N}(z_1 - z_2) > 1$. Since $\mathcal{O}_K$ is a PID, and both 2 and 3 are inert primes in K, we deduce that $\mathbf{N}(z_1 - z_2) \geq 5$. Hence $|z_1 - z_2| \cdot |z_1^\sigma - z_2^\sigma| \geq \sqrt{5}$. As $|z_1^\sigma - z_2^\sigma| \leq 2$, we have $|z_1 - z_2| \geq \frac{\sqrt{5}}{2}$. This contradicts with our assumption. $\square$

Proposition 2.4. *Suppose $z_1, z_2 \in S$. Then $|z_1 - z_2| \geq \frac{\sqrt{5}-1}{2}$.*

Proof. With no loss of generality we assume $|z_1 - z_2| < \frac{\sqrt{5}-1}{2}$. Then from the above lemma we know that $z_1 - z_2$ is a unit in K. Hence $|z_1 - z_2| = (\frac{\sqrt{5}-1}{2})^j$ for some $j \geq 0$. We need to show that $j \leq 1$.

Suppose to the contrary that $j \geq 2$, then $|z_1^\sigma - z_2^\sigma| \geq \frac{3+\sqrt{5}}{2}$. This gives rise to a contradiction as $|z_1^\sigma - z_2^\sigma| \leq 2$. □

Proposition 2.5. $\min_{z' \in \mathcal{S} - \{z\}} |z - z'| \leq 1$.

Proof. It suffices to show that at least one of $z + e^{\pi ij/5}$ is in S for $0 \leq j \leq 9$. Note that by definition of S this is equivalent to $\left|z^\sigma + \sigma(e^{\pi ij/5})\right| \leq 1$. Note that the inequality holds for $z = 0$ and all $0 \leq j \leq 9$. Now assume that $z \neq 0$. Then we may choose j' such that the argument of z^σ and $e^{\pi ij'/5}$ differs by no greater than $\pi/10$. Thus

$$\begin{aligned}
\left|z^\sigma - \sigma(e^{\pi ij'/5})\right|^2 &= |z^\sigma|^2 + 1 - 2\mathrm{Re} z^\sigma e^{-\pi ij'/5} \\
&\leq |z^\sigma|^2 + 1 - 2|z^\sigma| \cos(\pi/10) \\
&= (|z^\sigma| - \cos(\pi/10))^2 + \sin^2(\pi/10) \\
&\leq \cos^2(\pi/10) + \sin^2(\pi/10) = 1.
\end{aligned}$$

□

Proof of Theorem 2.2. From Proposition 2.5 we know that the minimum distance is always less than or equal to 1. From Proposition 2.4 it follows that the minimum distance is greater than $\frac{\sqrt{5}-1}{2}$. By Lemma 2.3, the minimum distance is a real unit of K in the interval $[\frac{\sqrt{5}-1}{2}, 1]$. Thus it takes values only in $\left\{\frac{\sqrt{5}-1}{2}, 1\right\}$. □

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